ON PROJECTIONS AND SIMULTANEOUS EXTENSIONS*

BY

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ABSTRACT

Let X_1, X_2 be subspaces of a completely regular space X. The bounded linear extension of $C(X_1 \cap X_2)$ into $C(X)$ are related to the projections of norm < 3 from $C(X_1) + C(X_2)$ onto $C(X)$.

1. **Introduction.** $C(X)$ denotes the Banach space of all bounded continuous real-valued functions on a topological space X, with the supremum norm. If B is a subspace of X, *a simultaneous extension* is a linear operator E from $C(B)$ to $C(X)$, such that for each f in $C(B)$, Ef is an extension of f. If R denotes the restriction operator of $C(X)$ to $C(B)$, then a simultaneous extension is a linear right inverse of R.

When a bounded simultaneous extension exists, $C(B)$ is isomorphic to the subspace $EC(B)$ of $C(X)$, and $P = ER$ is a projection (all "projections" in this paper are linear and bounded) of $C(X)$ onto this subspace.

If X is metric and B is closed in X, then there exists a simultaneous extension E of $C(B)$ to $C(X)$ with norm 1 [2]. In the general case a bounded simultaneous extension may fail to exist: Let $X = \beta N$ (the Stone-Čech compactification of the discrete sequence N), and $B = \beta N - N$. As proved in [1], $C(\beta N - N)$ is not isomorphic to a direct factor of $C(\beta N)$.

Corson and Lindenstrauss [4], found recently, for every $k \geqq 1$, a pair $B \subset X$ of compact Hausdorff spaces, such that there is a simultaneous extension of $C(B)$ to $C(X)$ with norm k, but no one with smaller norm.

Another simple relation between projections and simultaneous extensions was observed by Dean [3]: Let $C_0(X,B)$ denote the subspace of $C(X)$ of functions vanishing on B. If E is a bounded simultaneous extension of $C(B)$ to $C(X)$, then $I - ER$ is a projection of $C(X)$ onto $C_0(X,B)$. If R has a bounded (not necessarily linear) right inverse Q on $C(B)$ (e.g. when B is closed and X is normal—by Tietze's theorem), the converse is also true: If P is such a projection, define $E = Q - PQ$. E does not depend on the choice of Q (if Q' is another right inverse of R, then $(Q - PQ) - (Q' - PQ') = (Q - Q') - P(Q - Q') = 0$), and is a bounded linear extension.

In this paper we study a less immediate relation between projections and simultaneous extensions: Suppose we have two pairs: $B_1 \subset X_1$, $B_2 \subset X_2$ and a homeo-

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morphism h of B_1 onto B_2. We can "paste" the spaces X_1 and X_2 along the B_i by identifying all the points s in B_1 with the corresponding hs in B_2, the quotient space X having the quotient topology. $C(X)$ is naturally identified as a subspace of $C(X_1) \oplus C(X_2)$. The theorem relates projections of $C(X_1) \oplus C(X_2)$ onto $C(X)$ to simultaneous extensions from $C(B)$ to $C(X)$, where B is the image of the B_i under the quotient map. This is done for the case where the B_i are closed and nowhere dense, the X_i—completely regular, and there exist norm preserving extensions Q_i (not necessarily linear) of $C(B_i)$ to $C(X_i)$ (R_iQ_i is the identity on $C_i(B_i)$) $(i = 1,2)$.

2. THEOREM. (In the conditions specified above) *if P is a projection of $C(X_1) \oplus C(X_2)$ onto $C(X)$ and $\| P \| < 3$, then there is a simultaneous extension E of $C(B)$ to $C(X)$, with $\| E \| < (\| P \| - 1)/(3 - \| P \|)$.*

Proof. For f in $C(B)$ define $W_1 f$ as the restriction of $P(0 \oplus Q_2 f)$ to X_1 ($f_1 \oplus f_2$ denotes the function which is f_1 on X_1 and f_2 on X_2). $W_1 f$ is independent on the choice of Q_2—if Q_2' is another extension, then $P(0 \oplus Q_2 f) - P(0 \oplus Q_2' f) = P[0 \oplus (Q_2 - Q_2')f] = 0 \oplus (Q_2 - Q_2')f$ vanishes on X_1. W_1 is a linear operator from $C(B)$ to $C(X_1)$; $R_1 W_1$ is a linear operator from $C(B)$ into itself. Define W_2 symmetrically. For each f in $C(B)$ we have:

(1) $\quad (R_1 W_1 + R_2 W_2)f = R[P(Q_1 f \oplus 0) + P(0 \oplus Q_2 f)] = R(Q_1 f \oplus Q_2 f) = f.$

We shall give now some bounds for $W_i f$: Let f be in $C(B)$ with $\|f\| \leqq 1$, and x in B. Let $\varepsilon > 0$ be arbitrary. In the open set $U = \{s \in X_1 - B; |Q_1 f(s) - f(x)| < \varepsilon\}$ choose a point t that satisfies also: $|P(Q_1 f \oplus 0)(t) - P(Q_1 f \oplus 0)(x)| < \varepsilon$. Take a function g in $C(X)$ such that $0 \leqq g \leqq 1$, $g(t) = 1$ and g vanishes out of U. Consider the function $F = (-Q_1 f \oplus Q_2 f) + [1 + f(x)]g$ which belongs to $C(X_1) \oplus C(X_2)$ and satisfies $|F| < 1 + \varepsilon$

$$\cdot (PF)(t) = (Q_1 f \oplus Q_2 f)(t) - 2P(Q_1 f \oplus 0)(t) + [1 + f(x)]g(t) =$$
$$1 + 2f(x) - 2W_1 f(x) + [Q_1 f(t) - f(x)] + 2[P(Q_1 f \oplus 0)(t) - P(Q_1 f \oplus 0)(x)]$$

by the choice of t, we have: $(1 + \varepsilon)\| P \| \geqq (PF)(t) > 1 + 2f(x) - 3\varepsilon - 2W_1 f(x)$, and as ε was arbitrary, we can conclude (by symmetry) that

(2) $\quad W_i f(x) \geqq f(x) - \frac{1}{2}(\| P \| - 1)$ for all $x \in B; f \in C(B)$ with $|f| \leqq 1$ and

$$i = 1,2.$$

Combining (1) and (2), we get also the upper bounds: $W_i f(x) < \frac{1}{2}(\| P \| - 1)$.

A very similar method gives us the same upper bound for $x \in X_i - B$: Take $g \in C(S)$ such that $0 \leqq g \leqq 1$, $g(x) = 1$ and g vanishes out of $\{s \in X_1 - B; |Q_1 f(s) - Q_1 f(x)| < \varepsilon\}$. Consider the function $F = (-Q_1 f \oplus Q_2 f) + [1 + f(x)]g$. $|F| < 1 + \varepsilon$ and $(PF)(x) = -(Q_1 f \oplus Q_2 f)(x) + 2W_1 f(x) \ +$

$+2g(x)=1+2W_1f(x)$, hence $W_1f(x)\leqq\frac{1}{2}(\|P\|-1)$ whenever $|f|\leqq 1$. Using $(-f)$ instead of f, we get:

(3) $|W_if(x)|\leqq\frac{1}{2}(\|P\|-1)$ for all $x\in X; f\in C(B)$ with $|f|\leqq 1$; $i=1,2$.

If $\|f\|=1$, $\sup\{|W_if(x)|; x\in B\}\geqq\sup\{|f(x)|; x\in B\}-\frac{1}{2}(\|P\|-1)$ $=\frac{1}{2}(3-\|P\|)$ (by (2)), combining these results we get:

(4) $0<\frac{1}{2}(3-\|P\|)<\|R_iW_i\|<\|W_i\|<\frac{1}{2}(\|P\|-1)<1$.

These bounds imply that R_iW_i are isomorphisms of $C(B)$ into itself. We shall prove now that they are onto: Let f be in $C(B)$. Consider the sequence: $f_0=0$, and for $n>0$, $f_n=f_{n-1}+(R_1W_1)(f-f_{n-1})$. All the f_n are evidently in $R_1W_1C(B)$. We prove by induction that $f-f_n=(R_2W_2)^nf$—this is obvious for $n=0$, and for $n>0$: $f-f_n=(f-f_{n-1})-(R_1W_1)(f-f_{n-1})=(R_2W_2)(f-f_{n-1})=(R_2W_2)^nf$ by ((1)). This implies that the f_n converge uniformly to f, and as $R_1W_1C(B)$ is closed, $f\in R_1W_1C(B)$. By symmetry we have also $R_2W_2C(B)=C(B)$.

As R_iW_i is one-to-one, R_i must be one-to-one on $W_iC(B)$, and this establishes a simultaneous extension $E_i=R_i^{-1}$ of $C(B)$ to $W_iC(B)\subset C(X_i)$.

$Ef=E_1f\oplus E_2f$ is a simultaneous extension of $C(B)$ to $C(X)$, its norm is, by (4), not larger than $(\|P\|-1)/(3-\|P\|)$. Q.E.D.

3. Remark. 1) If B is not empty, then necessarily $\|P\|\geqq 2$. If $\|P\|=2$, then the extension E is norm preserving.

2) If we have bounded simultaneous extensions E_i of $C(B)$ to $C(X_i)$, a projection P of norm $1+2\|E_1\|\|E_2\|/(\|E_1\|+\|E_2\|)$ exists: for $f\oplus g$ in $C(X_1)\oplus C(X_2)$, define:

$$P(f\oplus g)=[f+\{\|E_2\|/(\|E_1\|+\|E_2\|)\}E_1(R_1f-R_2g)]$$

$$\oplus[g+\{\|E_1\|/(\|E_1\|+\|E_2\|)\}E_2(R_2g-R_1f)].$$

If we have $\|E_1\|=1$, then $\|P\|=1+2\|E_2\|/(1+\|E_2\|)$ hence $\|E_2\|=(\|P\|-1)/(3-\|P\|)$. This shows that the bound in the theorem is the best.

3) The projection in 2) can be defined also when $\|E_2\|=\infty$, and we get a projection of norm $1+2\|E_1\|$. If $\|E_1\|=1$ (e.g. $X_2=\beta N$, $B_2=\beta N-N$, $X_1=[\beta N-N]\times[0,1]$ and $B_1=[\beta N-N]\times\{0\}$), we get a projection of norm 3, but there is no bounded simultaneous extension of $C(B)$ to $C(X)$. "Reflecting" by an open and closed subset of $\beta N-N$, we get an example where a projection of norm 3 exists, but there is no bounded simultaneous extension of $C(B)$ to any of the $C(X_i)$.

4) In the symmetric case—when h can be extended to a homeomorphism H of X_1 onto X_2, we get easily a better extension: $W_1 f + W_2 f H$. Its norm is $\| P \| - 1$. Conversely, for an extension E, the projection defined in (2) has norm $\| E \| + 1$.

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